# Mathematics Methods

Unit 3

## Exponential functions

### 1. Exponential functions

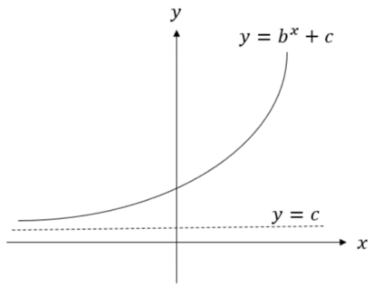
#### (a) General

Exponential function is in the form of $ab^x + c$.
Ex:

- $5^x$
- $2e^x + 2$
- $2^{-x}$

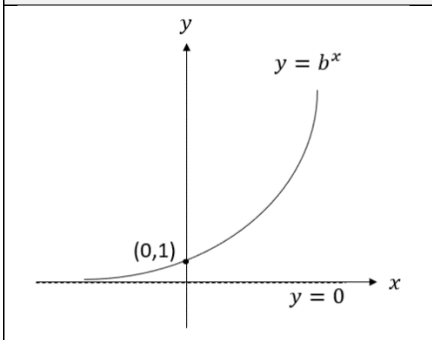
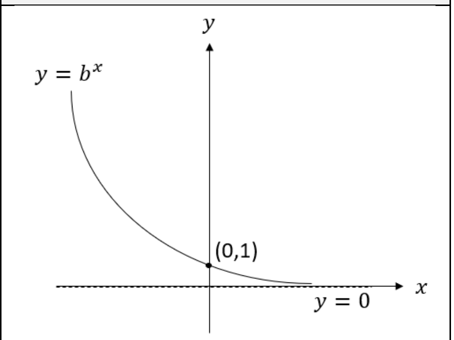
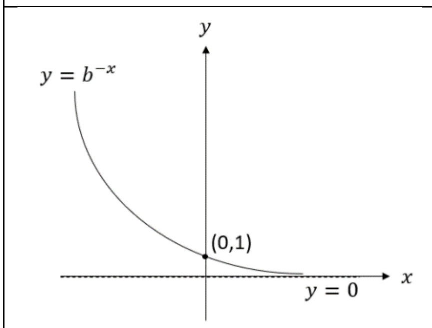
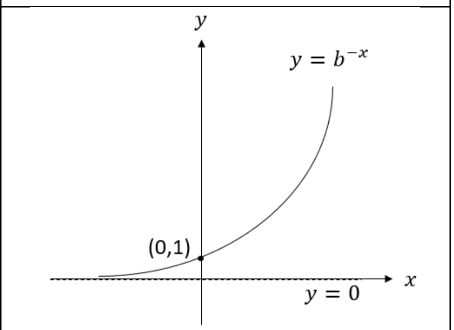
Characteristics of graphs of exponential function:

- Graph is asymptotic to the $x$-axis as $x$ approaches infinity
  Horizontal asymptote is $y = c$ (general graph horizontal asymptote is $y = 0$)

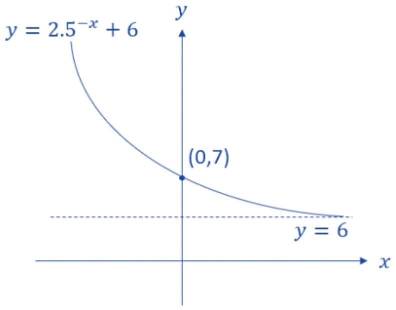



- Domain are real numbers
- Graph is smooth and continuous
- $y$-intercept at $(0, a + c)$ (general graph $y$-intercept at (0,1))

General exponential graph $y = b^x$:

| $b > 1$ | $0 < b < 1$ |
|---|---|
|  |  |
|  |  |

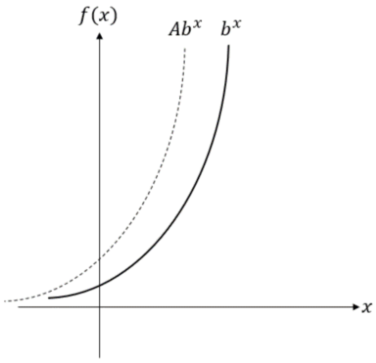
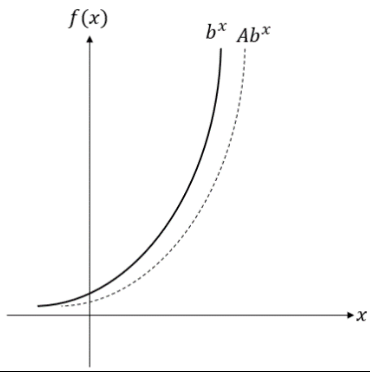
Example:
Sketch the graph for $g(x) = 2.5^{-x} + 6$ and label all the important features of the graph.

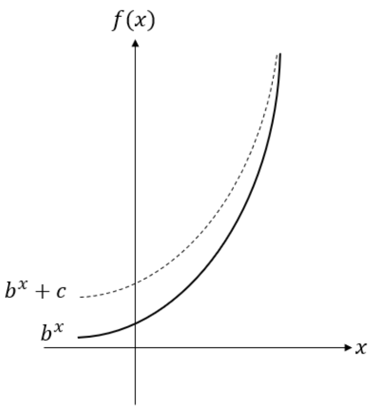
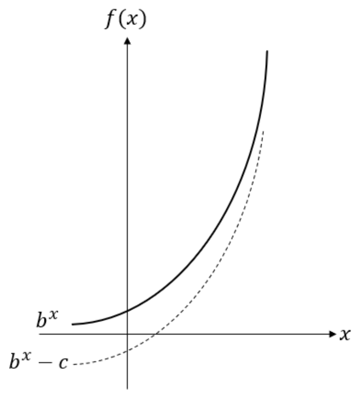



**(b) Transformations**

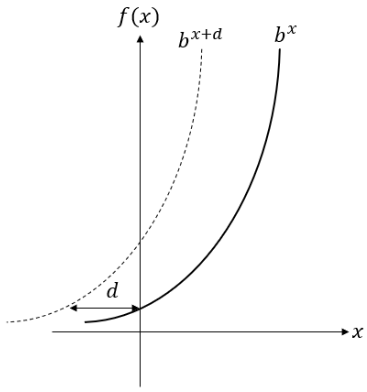
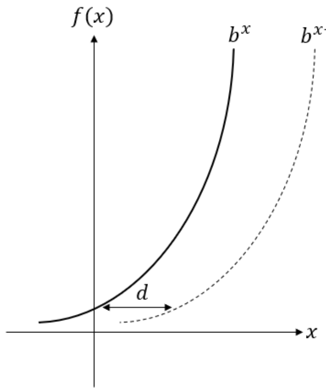
Given an exponential function as:

$$f(x) = Ab^{kx} + C$$

| **Effect of $A$** | |
|---|---|
| $\lvert A\rvert > 1$<br>$f(x)$ is vertically stretched | $0 < \lvert A\rvert < 1$<br>$f(x)$ is vertically compressed |
| Ex:  | Ex:  |

| **Effect of $C$** | |
|---|---|
| $+C$<br>$f(x)$ is vertically shifted upwards | $-C$<br>$f(x)$ is vertically shifted downwards |
| Ex:  Equation of horizontal asymptote is $y = c$ | Ex:  Equation of horizontal asymptote is $y = -c$ |

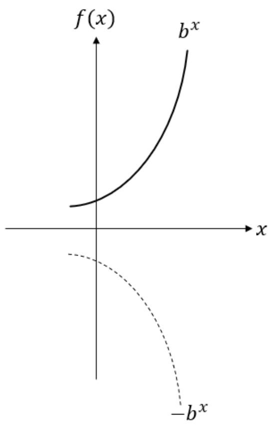
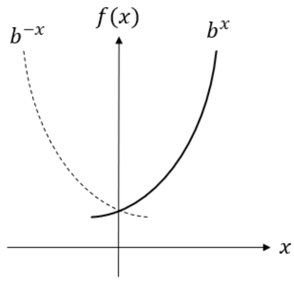
**Effect of $k$**

| $k + d$<br>$f(x)$ is horizontally shifted to left | $k - d$<br>$f(x)$ is horizontally shifted to right |
|---|---|
| Ex:  | Ex:  |





**Effect of $-$**

| $-b^x$<br>$f(x)$ is reflected along the horizontal axis | $b^{-x}$<br>$f(x)$ is reflected along the vertical axis |
|---|---|
| Ex:  | Ex:  |





**2. Euler's number**

Definition: a mathematical constant represented by $e$ named after a Swiss mathematician, Leonhard Euler

$$e = 2.7182818284\ldots \approx 2.71828$$

**(a) Derivation of Euler's number using the expression $\lim_{n \to 0} (1 + \frac{1}{n})^n$**

Let $f(x) = (1 + \frac{1}{n})^n$

| $n$ | 1 | 2 | 3 | … | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ | $10^7$ | $10^8$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $*f(x)$ | 2 | $\frac{9}{4}$ | $\frac{64}{27}$ | … | 2.70481 | 2.71692 | 2.71815 | 2.71827 | 2.71828 | | |

*$*f(x)$ in 5 decimal places*

∴ As $n$ approaches infinity, $e \approx 2.71828$

Variations:

| Expression | Euler's number expression |
|---|---|
| $(1+\frac{k}{n})^n$ | $e^k$ |
| $(1+\frac{1}{kn})^n$ | $e^{\frac{1}{k}}$ |

Example 1:

Given that $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$, evaluate $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n}$.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n} = e^2$$

Example 2:

Evaluate $\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{2n}$ given that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n} = e^2$ and that $\lim_{n\to\infty}(1+\frac{1}{kn})^n = e^{\frac{1}{k}}$.

$$\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{2n} = e^{\frac{2}{2}}$$
$$= e$$

**(b) Derivation of Euler's number through sum of infinite series**

Let $y = \frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+.\frac{1}{6!}+\frac{1}{7!}+\frac{1}{8!}+\frac{1}{9!}\ldots..$

$= 2.71828$ (sum of first 10 terms)

Alternatively,

$$\sum_{n=0}^{\infty}\frac{1}{n!} = e$$

Example 1:

Given that $e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$ Show that $e \approx 2.7182$ by substituting $x = 1$.

Using first 7 terms,
Sub $x = 1$,

$$e^1 = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\frac{1}{7!}$$
$$e = 2.71825$$
$$\approx 2.7182$$

Example 2:

Given that $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \ldots$. Deduce a general sum to infinite function that is equivalent.

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!}x^k$$

**3. Non-continuous exponential decay**

**Additional info.*

| **Exponential growth** | **Exponential decay** |
|---|---|
| Formula: $y = a(1+b)^t$ | Formula: $y = a(1-b)^t$ |
| $y$: the value after $t$,time passed (final value)<br>$a$: initial value<br>$b$: decay factor<br>$t$: time passed | |

Example 1:

The national park estimates that there are 5,000 tigers available. As illegal deforestation and poaching is prevalent in the country. The number of tigers is expected to fall at $1.7\%$ annually with some assumptions. What is the number of the tiger population after 2 years.

$$\begin{aligned} y &= a(1-b)^t \\ &= 5000(1-0.017)^2 \\ &= 4831.445 \\ &\approx 4831 \end{aligned}$$

Example 2:

Jack deposited \$1,200 into her bank account. The interest rate is 2% compounded annually. What is the amount is Jack's bank after two years?

$$\begin{aligned} y &= a(1+b)^t \\ &= 1200(1+0.02)^2 \\ &= \$\,1248.48 \end{aligned}$$

Example 3:

The population of rabbits is increasing at $18\%$ every two years. Given that the initial population of rabbits is 3,000. What is the population of rabbits after six years?

$$\begin{aligned} y &= a(1+b)^{\frac{t}{2}} \\ &= 3000(1+0.18)^{\frac{6}{2}} \\ &= 4929.1 \\ &\approx 4929 \end{aligned}$$

| 3. | **Continuous exponential growth/ decay** |
|---|---|
| | Characteristics of:<br>• Exponential growth: $y$ increases as $x$ increases<br>• Exponential decay: $y$ decreases as $x$ increases<br><br>Formula:<br>$$A = A_o e^{kt}$$<br>$A:$ the value after $t$,time passed (final value)<br>$A_o:$ initial value<br>$k:$ growth rate ($k > 0$: exponential growth, $k < 0$: exponential decay)<br>$t:$ time passed<br><br>Rate of change<br>$$\frac{dA}{dt} = kA$$ |
| | Example 1:<br>After the re-snap, half of the population of humans revives continuously such that $\frac{dp}{dt} = 5.2\ t$ in which $t$ is the population of humans after $t$ hours. How long does it take for the population of human to be 7.7 billion? Give your answer in 3 significant figures<br><br>$7.7 = (7.7 \div 2)e^{5.2\ t}$<br>$7.7 = 3.85e^{5.2t}$<br>$e^{5.2t} = 2$<br>$5.2t = \frac{log_{10} 2}{log_{10} e}$<br>$5.2t = 0.69314718$<br>$t = 0.133\ hours$<br>Example 2:<br>The population of uncontrolled rats is initially 400. The estimated growth rate of rates after $n$, months is given by $400e^{0.12n}$. Estimate the population of rats after 2 months.<br><br>$P = 400e^{0.12n}$<br>$= 400e^{0.12(2)}$<br>$= 508.5$<br>$\approx 509$<br><br>Example 3:<br>The initial population of bees of 5000 is decreasing at rate of $\frac{dy}{dx} = -0.9\ y$ in which $y$ is the population of bees while $x$ is the years from initial population. Find the population of bees after 5 years.<br><br>$P = P_o e^{kx}$<br>$= 5000e^{-0.9(5)}$<br>$= 55.545$<br>$\approx 56$ |

Example 4:
The information box below conveys about Sumatran tigers.

> The Sumatran tiger is a Panthera tigris sondaica population in the Indonesian island of Sumatra. This population was listed as Critically Endangered on the IUCN Red List in 2008, as it was estimated at 441 to 679 individuals, with no subpopulation larger than 50 individuals and a declining trend. (Wikipedia)

Find the rate of extinction of Sumatran tiger after two years given that the rate of extinction can be given by $\frac{dP}{dt} = -0.9\,P$ where $t$ is the time passed in years. Take the initial number of tigers as the mean of the estimated range in year 2008.

$$\frac{dP}{dt} = -0.9\,P$$
$$P = P_o e^{kt}$$

$$\frac{dP}{dt} = -0.9P_o e^{kt}$$
$$= -0.9\left(\frac{441+679}{2}\right)e^{-0.9(2)}$$
$$= -83.31$$

**END**